Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 27, 2002*
Accepted: *April 22, 2002*

Sigma-model soliton intersections from exceptional calibrations

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Abstract: A first-order 'BPS' equation is obtained for 1/8 supersymmetric intersections of soliton-membranes (lumps) of supersymmetric $(4+1)$-dimensional massless sigma models, and a special non-singular solution is found that preserves 1/4 supersymmetry. For 4-dimensional hyper-Kähler target spaces (HK_4) the BPS equation is shown to be the low-energy limit of the equation for a Cayley-calibrated 4-surface in $\mathbb{E}^4 \times HK_4$. Similar first-order equations are found for stationary intersections of Q-lump-membranes of the massive sigma model, but now generic solutions preserve either 1/8 supersymmetry or no supersymmetry, depending on the time orientation.

Keywords: Sigma Models, Solitons Monopoles and Instantons, M-Theory, p-branes.

Contents

1. Introduction

Supersymmetric sigma models, in dimension (1+1) or above, are known to have a variety of supersymmetry-preserving, and hence stable, soliton solutions, supported by a combination of topological and Noether charges. It is natural to consider the maximally-supersymmetric models, with a total of 8 supersymmetries, because sigma models with fewer supersymmetries can always be embedded in some model with maximal supersymmetry, for which the target space is automatically hyper-Kähler (HK), although not necessarily irreducible.[1] In this context the fraction of supersymmetry preserved by a supersymmetric solution may be[2] 1/2, 1/4 or 1/8, and this fraction provides a convenient partial characterisation of the solution; roughly speaking, the lower the fraction the more complicated the solution is.

The basic sigma-model soliton is the 1/2 supersymmetric sigma-model lump (see, for example, [1]), which is a particle-like solution of massless $(2+1)$-dimensional HK sigma models, but a string-like solution (the lump-string) of $(3+1)$-dimensional models, and so on up to a 3-brane in the maximal dimension of $(5+1)$. The sigma-model lump is 'basic' in two respects. Firstly, it gives rise to the 1/2 supersymmetric kink soliton of the massive $(1+1)$-dimensional sigma model obtained by non-trivial dimensional reduction on

[1] E.g. a Calabi-Yau 3-fold $\mathcal{M}$ is a submanifold of the 12 real dimensional target space $\mathcal{M} \times T^6$, which has holonomy $SU(3) \subset Sp_3$, and therefore yields a sigma model with maximal supersymmetry.

[2] The fraction 3/8 might be possible, in principle, but no example is known.

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S^1 [2, 3]. Secondly, it is the main ingredient in the construction of solutions preserving only 1/4 supersymmetry via 'intersections'. The first example of this was a non-singular 1/4-supersymmetric intersection of two kink domain walls in massive $(1+1)$-dimensional HK sigma models of quaternionic dimension $n \geq 2$ [4]. Another example, which is also possible for $n = 1$, is the kink-lump of massive $(3+1)$ dimensional HK sigma models [5]; this is a non-singular solution in which a lump-string ends on a kink-domain-wall, thus imitating the physics of D2-branes in type-IIA superstring theory.

A IIA superstring ending on a D2-brane can be viewed, from an 11-dimensional point of view, as a pair of intersecting supermembranes. Similarly, as we demonstrate here, the kink-lump of the massive $(3+1)$-dimensional sigma model lifts to a pair of 1/4 supersymmetric intersecting lump-membranes of the massless $(4+1)$-dimensional HK sigma model. However, our main interest in this paper is intersections of these lump-membranes that preserve the minimal fraction of 1/8 supersymmetry. We obtain a first-order 'BPS' equation for intersections of lump-membranes that is solved by our explicit 1/4 supersymmetric configuration but for which the generic solution preserves only 1/8 supersymmetry. Of course, there may be no generic solution that satisfies the required boundary conditions in any given model, but we propose a particular model that we believe will have 1/8 supersymmetric solutions of the desired type.

A trivial dimensional reduction of the massless $(4+1)$-dimensional HK sigma model to a massless $(3+1)$-dimensional model reduces the lump-membranes to lump-strings. It also reduces the BPS equation for 1/8 supersymmetric intersections of lump-membranes to a previously proposed BPS equation for 1/4 supersymmetric intersections of lump-strings [6]. Another way to interpret this fact is to say that any 1/8 supersymmetric configuration of lump-membranes will degenerate to a 1/4 supersymmetric solution in regions in which it becomes independent of one of the four space coordinates, although this 1/4 supersymmetric solution is of an essentially different type to the pair of intersecting membranes mentioned above because it is a solution of the sigma-model equations that is intrinsically 3-dimensional rather than 4-dimensional. If this solution becomes independent of *another* of the four space coordinates then it reduces to the 1/2 supersymmetric, and intrinsically 2-dimensional, lump-membrane.

A *non-trivial* dimensional reduction of the massless $(4+1)$-dimensional HK sigma model (when this is possible) yields a massive $(3+1)$-dimensional HK sigma model, which admits static 1/2 supersymmetric kink-membranes and stationary, charged, 1/4 supersymmetric Q-lump-strings [7, 5]. Apart from the 1/4 supersymmetric kink-lump mentioned above, and its stationary Q-kink-lump counterpart [5], these massive models can also be expected to have 1/8 supersymmetric intersections; for example, a BPS equation for 1/8 supersymmetric intersecting Q-lump-strings is known [6]. If the HK target space is 4-dimensional, as it is for the models we consider here, then any solution of the massive $(3+1)$-dimensional sigma model is also a solution of the *same* massive model in $(4+1)$-dimensions[3], but additional solutions are possible in the higher dimension. In particular,

[3]This need not be true for higher-dimensional HK target spaces because in this case the $(3+1)$-dimensional model allows a more general potential than the $(4+1)$-dimensional model; precisely this fact was exploited in [4].

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our new BPS equation for 1/8 supersymmetric intersections of static lump-membranes of the massless $(4+1)$-dimensional model extends to a pair of BPS equations for 1/8 supersymmetric intersections of stationary Q-lump-membranes. A curious feature of generic solutions of these equations is that *supersymmetry is preserved for only one time orientation*, despite the time-reversal invariance of the full second-order field equations.

Supersymmetric sigma models in $(5+1)$ dimensions with 4-dimensional HK target spaces (quaternionic dimension 1) provide a low energy effective description of the dynamics of an M5-brane in a spacetime of the form $\mathbb{E}^{(1,5)} \times HK_4 \times S^1$ in the special case of vanishing worldvolume 3-form field strength and fixed position on S^1; this truncation preserves the 8 supersymmetries that are left unbroken by the combination of the M5-brane and the HK background spacetime. Given that the $\mathbb{E}^{(1,5)}$ factor of the background is identified as the fivebrane vacuum, the fivebrane dynamics is then governed by an action for maps from $\mathbb{E}^{(1,5)}$ to HK_4, for which the low-energy limit is the (1,0)-supersymmetric $(5+1)$-dimensional sigma model with HK_4 as its target space (see [8] for a review).

The 1/2 supersymmetric lump solitons of such sigma models can be interpreted as smooth 3-brane intersections of the original M5-brane with another M5-brane that is wrapped on a finite area holomorphic 2-cycle of HK_4 [9, 5]. Actually, since the intersection is smooth, one may consider the two intersecting M5-branes in this background as a single Kähler-calibrated M5-brane [10]. In this M-theory context it is natural to suppose that intersections of sigma-model solitons will also have an interpretation as calibrations. However, whereas Kähler-calibrated surfaces are solutions of equations that are homogeneous in derivatives, as are the BPS equations for sigma-model solitons and their intersections, the equations of other types of calibrated surfaces are *inhomogeneous in derivatives* and therefore have no obvious interpretation as BPS equations in field theory.

Nevertheless, since a calibrated p-surface solves the equations of motion for a p-brane, any solutions of the 'linearized' calibration equation (obtained by linearizing in derivatives) must solve the 'linearized' p-brane equations. In flat space these 'linearized' equations have no interesting solutions, not surprisingly since the 'linearized' equations are truly linear in this case, being linear in fields as well as derivatives. But in non-flat backgrounds the 'linearized' calibration equations will still be non-linear in fields and hence may have interesting solutions. In fact, we shall show that both the BPS equation of [6] for 1/4 supersymmetric intersecting sigma-model solitons and the new BPS equation presented here for 1/8 supersymmetric intersections are 'linearizations' of the equations for, respectively, associative and Cayley calibrations. Thus, intersecting HK sigma-model solitons can be understood in terms of exceptional calibrations of a single M5-brane of M-theory. This result is reminiscent of the interpretation of junctions of MQCD domain walls as Cayley calibrated M5-branes [11], but we are not aware of any direct connection to the results reported here.

2. Supersymmetric HK sigma models

We begin with a brief discussion of the models to be considered. Let X^I $(I = 1, \ldots, 4n)$ be the scalar fields. Omitting fermions, the massless supersymmetric sigma model in $(5+1)$

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dimensions has the action

$$S_6[X] = \frac{1}{2}\int dtd^5x\left[|\dot{X}|^2 - |\nabla X|^2\right], \tag{2.1}$$

where the overdot indicates differentiation with respect to the time coordinate t, and ∇ is the partial derivative with respect to the five cartesian space coordinates. As the target space is necessarily HK, there exist three complex structures $\mathbf{I} = (I_1, I_2, I_3)$, with matrix entries $(I_i)^I{}_J$ $(i = 1, 2, 3)$, obeying the algebra of the quaternions

$$I_i I_j = -\delta_{ij}\mathbb{I} + \varepsilon_{ijk} I_k\,, \tag{2.2}$$

where $\mathbb{I}$ is the identity matrix. The triplet of complex structures is associated with the triplet of (closed) Kahler 2-forms

$$\mathbf{\Omega}_{IJ} = g_{IK}\mathbf{I}^K{}_J\,. \tag{2.3}$$

The massless (5+1)-dimensional model can of course be trivially dimensionally reduced to any lower spacetime dimension, but given that the target space admits a U(1) Killing vector field (KVF) k, there is also the possibility of a non-trivial dimensional reduction, achieved by setting

$$\partial_5 X^I = m\,k^I\,, \tag{2.4}$$

where m is a mass-parameter. This reduction will preserve all 8 supersymmetries if k is triholomorphic, which is the requirement of vanishing $\mathcal{L}_k\mathbf{\Omega}$, where $\mathcal{L}_k$ is the Lie derivative with respect to the vector field k. The resulting maximally supersymmetric $(4+1)$-dimensional sigma model has the action (again supressing fermions)

$$S_5[X] = \frac{1}{2}\int dtd^4x\left[|\dot{X}|^2 - |\partial X|^2 - m^2|k|^2\right], \tag{2.5}$$

where ∂ indicates differentiation with respect to the *four* space coordinates. For $m \neq 0$ there is a positive scalar potential, given by the norm squared of the triholomorphic KVF k [12]; this potential vanishes at the isolated fixed points of k.

Although many of our results on HK sigma models are valid for target spaces of arbitrary quaternionic dimension n, we shall concentrate here on the $n = 1$ case because of its M-theory interpretation and the connection to exceptional calibrations. Specifically, we shall consider 4-dimensional HK manifolds with a triholomorphic U(1) KVF. In coordinates $(\varphi, \mathbf{X})$ for which this KVF is

$$k = \frac{\partial}{\partial\varphi}\,, \tag{2.6}$$

the metric is

$$ds^2 = Ud\mathbf{X}\cdot d\mathbf{X} + U^{-1}(d\varphi + \mathbf{A}\cdot d\mathbf{X})^2 \tag{2.7}$$

where U is a harmonic function on $\mathbb{E}^3$ with isolated point singularities and $\mathbf{A}$ is a 3-vector potential such that $\mathbf{\nabla}\times\mathbf{A} = \mathbf{\nabla}\varphi$. If φ is identified with period 2π then completeness of the metric requires

$$U = a + \frac{1}{2}\sum_{k=1}^{N}|\mathbf{X} - \mathbf{X}_k|^{-1} \tag{2.8}$$

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for some number a, integer N, and a choice of N points $\mathbf{X}_k$ in $\mathbb{E}^3$, which are called the 'centres' [13]. The triplet of Kähler 2-forms is

$$\mathbf{\Omega} = \mathcal{D}\varphi d\mathbf{X} - \frac{1}{2}U d\mathbf{X} \times d\mathbf{X} \tag{2.9}$$

where we have supressed the wedge product of forms, and defined the covariant derivative

$$\mathcal{D}\varphi = d\varphi + \mathbf{A} \cdot d\mathbf{X}\,. \tag{2.10}$$

The coefficients of $\mathbf{\Omega}$ are φ-independent, so k is triholomorphic.

For these 4-dimensional HK target spaces the massless (5+1)-dimensional sigma model has the action

$$S_6[\varphi, \mathbf{X}] = \frac{1}{2}\int dt d^5x \left\{U^{-1}\left[(\mathcal{D}_t\varphi)^2 - |\mathcal{D}\varphi|^2\right] + U\left[|\dot{\mathbf{X}}|^2 - |\nabla\mathbf{X}|^2\right]\right\}\,. \tag{2.11}$$

The dimensional reduction ansatz (2.4) is now

$$\mathcal{D}_5\varphi = m\,, \qquad \partial_5\mathbf{X} = \mathbf{0}\,, \tag{2.12}$$

and this yields the $(4+1)$-dimensional action

$$S_5[\varphi, \mathbf{X}] = \frac{1}{2}\int dt d^4x \left\{U^{-1}\left[(\mathcal{D}_t\varphi)^2 - |\mathcal{D}\varphi|^2\right] + U\left[|\dot{\mathbf{X}}|^2 - |\partial\mathbf{X}|^2\right] - m^2U^{-1}\right\}. \tag{2.13}$$

Note that the potential m^2U^{-1} is non-negative and has zeros at the singularities of U; i.e., at the centres of the metric.

3. Lumps and Q-lumps, Kinks and Q-kinks

As a prelude to our results on intersecting sigma-model solitons we will briefly review some aspects of the solitons themselves. We start with the lump and Q-lump solitons of $(2+1)$-dimensional HK sigma models. The energy functional is

$$E = \frac{1}{2}\int d^2x \left\{|\dot{X}|^2 + |\partial_1 X|^2 + |\partial_2 X|^2 + m^2|k|^2\right\}\,. \tag{3.1}$$

For $m = 0$ this is a massless sigma model while for $m \neq 0$ it is a massive one. Introducing a constant unit 3-vector $\mathbf{n}$, we may rewrite the energy functional as [7]

$$E = \frac{1}{2}\int d^2x \left\{|\dot{X} \mp mk|^2 + |\partial_1 X - \mathbf{n}\cdot\mathbf{I}\,\partial_2 X|^2\right\} \pm mQ + \mathbf{n}\cdot\mathbf{L} \tag{3.2}$$

where Q is the U(1) Noether charge associated to the symmetry generated by k,

$$Q = \int d^4x\, \dot{X}\cdot k\,, \tag{3.3}$$

and $\mathbf{L}$ is the topological 'lump' charge

$$\mathbf{L} = \int_{\mathbb{C}} f^*(\mathbf{\Omega})\,, \tag{3.4}$$

where f is the sigma-model map from the euclidean 2-space, viewed as the complex plane $\mathbb{C}$, and $f^*(\boldsymbol{\Omega})$ is its pullback. For fixed Q and $\mathbf{L}$ the energy is minimized by solutions of the equations

$$\dot{X}^I = \pm m k^I \tag{3.5}$$

and

$$\partial_1 X^I = (\mathbf{n}\cdot\mathbf{I})^I{}_J\, \partial_2 X^J\,. \tag{3.6}$$

The energy of solutions to these equations is given by

$$E = |\mathbf{L}| + m|Q|\,. \tag{3.7}$$

For $m = 0$ the sigma-model fields define a time-independent holomorphic map with respect to the complex structure $\mathbf{n}\cdot\mathbf{I}$. Given a 2-cycle of the HK manifold that is holomorphic with respect to this complex structure, there exist finite energy holomorphic maps from $\mathbb{C}$ to it. These are the sigma-model lump (or multi-lump) solitons; it can be shown that they preserve 1/2 supersymmetry. For $m \neq 0$, the holomorphic map is time-dependent but the energy density remains time-independent, so we now have non-static but *stationary* solitons. These are the Q-lumps, which can be shown to preserve 1/4 supersymmetry.

The simplest (HK) example of both lumps and Q-lumps is provided by a metric of the type described in the previous section with

$$U = \frac{1}{2}\left[\frac{1}{|\mathbf{X}-\mathbf{a}|} + \frac{1}{|\mathbf{X}+\mathbf{a}|}\right] \tag{3.8}$$

for unit vector $\mathbf{a}$. Because U preserves an SO(2) subgroup of the SO(3) rotation group acting on the 3-vector $\mathbf{X}$ there is a consistent truncation to a Kahler sigma model with S^2 target space parametrized by $(\varphi, A \equiv \mathbf{a}\cdot\mathbf{X})$ with $|A| \le 1$. For this truncated model,

$$U^{-1} = 1 - A^2\,, \tag{3.9}$$

and we can choose the vector potential $\mathbf{A}$ such that $\mathcal{D}\varphi = d\varphi$ [5]. The two singular points at $A = \pm 1$ are the north and south poles of the target 2-sphere, which is a holomorphic homology 2-sphere of the original HK target space. A sigma-model lump is obtained as a solution of the BPS equation (3.6) with $\mathbf{n} = \mathbf{a}$. The anti-lump is a solution of this equation for $\mathbf{n} = -\mathbf{a}$. All other solutions, including all solutions for $\mathbf{n} \neq \pm\mathbf{a}$, are supersymmetric 'BPS flows' but not ones corresponding to *finite* energy.

We now turn to the kinks and Q-kinks. These are solutions of the massive $(1+1)$-dimensional sigma model, for which the energy functional is

$$E = \frac{1}{2}\int dx\, \left\{|\dot{X}|^2 + |X'|^2 + m^2|k|^2\right\}, \tag{3.10}$$

where the prime indicates differentiation with respect to the one space coordinate. Introducing a constant v such that $v^2 < 1$, we may rewrite the energy functional as [2]

$$\begin{aligned} E = \frac{1}{2}\int dx\, \left\{|\dot{X} \mp m v k|^2 + |X' - m\sqrt{1-v^2}(\mathbf{n}\cdot\mathbf{I})k|^2\right\} \pm \\ \pm\, m v Q + m\sqrt{1-v^2}(\mathbf{n}\cdot\mathbf{K}) \end{aligned} \tag{3.11}$$

where $\mathbf{K}$ is the topological 'kink' charge

$$\mathbf{K} = \int_{\mathbb{R}} f^*(i_k \mathbf{\Omega})\,. \tag{3.12}$$

For fixed Q and $\mathbf{K}$, the energy is minimized by solutions of the first-order equations

$$\dot{X}^I = \pm m v k^I \tag{3.13}$$

and

$$(X^I)' = m\sqrt{1-v^2}\,(\mathbf{n}\cdot\mathbf{I})^I{}_J\, k^J\,. \tag{3.14}$$

When $v = 0$ we have a static kink and when $v \neq 0$ we have a stationary Q-kink; in either case the energy is

$$E = m\sqrt{|\mathbf{K}|^2 + Q^2}\,, \tag{3.15}$$

and both kink and Q-kink preserve 1/2 supersymmetry.

The simplest HK examples again occur for 4-dimensional multi-centre target spaces, for which (3.13) and (3.14) are equivalent to

$$\mathcal{D}_t\varphi = \pm m v\,, \qquad \dot{\mathbf{X}} = \mathbf{0}\,, \qquad \mathbf{X}' = m\sqrt{1-v^2}\,U^{-1}\mathbf{n} \tag{3.16}$$

for some unit vector $\mathbf{n}$. For special choices of $\mathbf{n}$, the solutions of these equations will be BPS flows with finite energy; these are the kinks and Q-kinks, which connect the vacua at the singularities of U. For the 2-centre case with U given by (3.8) we can solve the equations (3.16) by again considering the Kähler truncation; this shows that the finite energy BPS flows occur for $\mathbf{n} = \pm\mathbf{a}$.

In general, not every vacuum of the massive sigma model will be connected to every other one by a kink but the vacua must form a connected set. Note that BPS flows cannot cross for a *given* $\mathbf{n}$, but there is no such restriction on BPS flows corresponding to different choices of $\mathbf{n}$. For example, let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three mutually orthogonal unit 3-vectors and consider a 6-centre metric with

$$U = \frac{1}{2}\sum_{\mp}\left[\frac{1}{|\mathbf{X}\mp\mathbf{a}|} + \frac{1}{|\mathbf{X}\mp\mathbf{b}|} + \frac{1}{|\mathbf{X}\mp\mathbf{c}|}\right]. \tag{3.17}$$

This choice preserves a discrete subgroup of SO(3) that permutes the three axes of $\mathbb{E}^3$. In particular, there is a Z_2 subgroup that fixes $\mathbf{a}\cdot\mathbf{X}$ but interchanges $\mathbf{b}\cdot\mathbf{X}$ and $\mathbf{c}\cdot\mathbf{X}$. This is sufficient to ensure the consistency of the truncation to a Kähler sigma model with fields $(\varphi, A \equiv \mathbf{a}\cdot\mathbf{X})$, for which

$$U = \frac{1}{1-A^2} + \frac{2}{\sqrt{1+A^2}} \tag{3.18}$$

Although the BPS flow interpolating between $A = -1$ and $A = 1$ is now more complicated, it exists. By symmetry so do BPS flows interpolating between $\mathbf{X} = -\mathbf{b}$ and $\mathbf{X} = \mathbf{b}$ (for $\mathbf{n} = \mathbf{b}$), and between $\mathbf{X} = -\mathbf{c}$ and $\mathbf{X} = \mathbf{c}$ (for $\mathbf{n} = \mathbf{c}$).

Similar considerations apply to lumps of the (2+1)-dimensional massless sigma model; whenever there exists a massive sigma-model kink interpolating between two centres of the

HK_4 metric then there exists a holomorphic map from $\mathbb{C}$ to a homology 2-sphere with poles at the two centres. We will use this fact in the following section to argue that a $(4+1)$-dimensional supersymmetric HK sigma model with the 4-dimensional 6-centre target space just described admits 1/8 supersymmetric intersections of its lump-membrane solitons.

4. BPS equations for intersecting solitons

We now turn to *intersecting* solitons of the $(4+1)$-dimensional sigma model. The energy functional is

$$E = \frac{1}{2}\int d^4x \left\{|\dot{X}|^2 + |\boldsymbol{\nabla} X|^2 + |\partial_4 X|^2 + m^2|k|^2\right\} \tag{4.1}$$

where we have set $x = (\mathbf{x}, x^4)$ and $\partial = (\boldsymbol{\nabla}, \partial_4)$. For $m \neq 0$ this is a massive sigma model but we may set $m = 0$ to get the massless model. We now rewrite this energy functional as

$$E = \frac{1}{2}\int d^4x \left\{|\dot{X} \mp mk|^2 + |\partial_4 X^I - \mathbf{I}^I{}_J \cdot \boldsymbol{\nabla} X^J|^2\right\} \pm mQ + T \tag{4.2}$$

where T is the surface term

$$T = \frac{1}{2}\int d^4x\, \boldsymbol{\nabla} X^I \times \boldsymbol{\nabla} X^J \cdot \boldsymbol{\Omega}_{IJ} + \int d^4x\, \partial_4 X^I \boldsymbol{\nabla} X^J \cdot \boldsymbol{\Omega}_{IJ}\,. \tag{4.3}$$

The energy is therefore minimized, for boundary conditions that fix the value of Q and T, by solutions of the first order 'BPS' equations

$$\dot{X}^I = \pm m k^I \tag{4.4}$$

and

$$\partial_4 X^I = \mathbf{I}^I{}_J \cdot \boldsymbol{\nabla} X^J\,. \tag{4.5}$$

For sigma-model fields that are independent of x^1 the latter equation reduces to

$$\partial_4 X^I = (I_2)^I{}_J \partial_2 X^J + (I_3)^I{}_J \partial_3 X^J\,, \tag{4.6}$$

which is equivalent to the equation considered in the context of the $(3+1)$-dimensional HK sigma model in [6]. Because the three complex structures $\mathbf{I}$ obey the algebra of the quaternions, an analogous equation holds for configurations that are independent of any one of the four space coordinates.

When the target space is a 4-dimensional HK manifold of the type described in the introduction, the equation (4.4) becomes

$$\dot{\varphi} = \pm m\,, \tag{4.7}$$

which is trivially solved, while the BPS equation (4.5) becomes the set of equations

$$\boldsymbol{\nabla} \cdot \mathbf{X} = U^{-1}\mathcal{D}_4\varphi\,, \qquad \boldsymbol{\nabla} \times \mathbf{X} = U^{-1}\boldsymbol{\mathcal{D}}\varphi + \partial_4 \mathbf{X}\,. \tag{4.8}$$

For configurations that are independent of x^4 these equations reduce to

$$\boldsymbol{\nabla} \cdot \mathbf{X} = 0\,, \qquad \boldsymbol{\nabla} \times \mathbf{X} = U^{-1}\boldsymbol{\mathcal{D}}\varphi\,, \tag{4.9}$$

which describe intersecting lump-strings in the model obtained by a trivial reduction to (3+1) spacetime dimensions. If we further specialize to configurations that are independent of x^3 then we find, firstly, that X_1 and X_2 are harmonic functions on the 12-plane; for non-singular solutions with physical behaviour at infinity we must therefore set $X_1 = X_2 = 0$. We are then left with X_3 and φ which are required to satisfy

$$\partial_1 X_3 = -U^{-1}\mathcal{D}_2\varphi\,, \qquad \partial_2 X_3 = U^{-1}\mathcal{D}_1\varphi \tag{4.10}$$

These are the equations for the sigma-model lump in the form given in [5].

We now consider the interpretation of (4.5) or, more specifically, (4.8). Suppose that we have a solution that is asymptotically independent of, say, x^1 and x^2 as both x^3 and x^4 become large (for fixed x^1, x^2). Such a solution asymptotes to a lump-membrane in the 12-plane. One can now imagine a solution of this type that is symmetric under permutations of the four axes, in which case it will describe the intersection of six lump-membranes, one for each of the planes through the origin containing two of the four axes. It is possible that 1/8 supersymmetric solutions of (4.8) of this type will exist only for special models. We will see in the following section that a simple 2-centre model allows a special solution that can be interpreted as a non-singular intersection of two totally orthogonal membranes. A similar type of solution must exist for the 6-centre model described earlier with $\mathbf{n} = \mathbf{a}$, for the reasons explained there, but now one can contemplate a superposition with two similar solutions obtained from the BPS equations for $\mathbf{n} = \mathbf{b}$ and $\mathbf{n} = \mathbf{c}$. By choosing boundary conditions that preserve the permutation symmetry of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and that specify the appropriate fall-off of the fields away from any of the axes, one would expect to find a solution in which six lump-membranes intersect in the way just described, although we anticipate that this may be difficult to verify in practice.

5. An explicit intersecting lump solution

We will now exhibit a particular static solution of (4.8) for the 2-centre massless HK sigma model with U given by (3.8). We will show later that it preserves 1/4 supersymmetry rather than 1/8 supersymmetry, so it is not a generic solution. Nevertheless, it demonstrates that non-singular intersections of lump-membranes can occur. We first set

$$X_1 = A\,, \qquad X_2 = X_3 = 0\,, \tag{5.1}$$

to reduce the four equations to the two pairs of equations

$$\partial_1 A = U^{-1}\mathcal{D}_4\varphi\,, \qquad \partial_4 A = -U^{-1}\mathcal{D}_1\varphi \tag{5.2}$$

and

$$\partial_3 A = U^{-1}\mathcal{D}_2\varphi\,, \qquad \partial_2 A = -U^{-1}\mathcal{D}_3\varphi\,. \tag{5.3}$$

These equations become the kink-lump equations solved in [5] on non-trivial reduction to (3+1) dimensions. As explained there, we may now choose a gauge for which $\mathcal{D}\varphi = d\varphi$.

Noting that the potential U^{-1} is given by (3.9), we can easily solve (5.2) and (5.3). The solution is

$$A = \tanh \log |Z| \, , \qquad \varphi = \arg Z \tag{5.4}$$

for a function $Z(\zeta, \xi)$ that is holomorphic in the two complex variables

$$\zeta = x^2 + ix^3 \, , \qquad \xi = x^4 + ix^1 \, . \tag{5.5}$$

The simplest non-trivial choice for Z is

$$Z = \frac{c}{\zeta\xi} \tag{5.6}$$

for some complex number c. For fixed ζ we have a lump in the ξ-plane and hence a membrane parallel to the ζ-plane. Similarly, for fixed ξ we have a lump in the ζ-plane and hence a membrane parallel to the ξ-plane. Thus, the full non-singular solution represents a pair of intersecting membranes.

Confirmation of this interpretation can be had from an investigation of the energy. For the general solution of the form (5.4) the energy density is

$$\mathcal{E} = \left(1 + |Z|^2\right)^{-2} \left[|\partial_\zeta Z|^2 + |\partial_\xi Z|^2\right] . \tag{5.7}$$

For the particular solution (5.6) this becomes

$$\mathcal{E}(u, v) = \frac{4|c|^2 (u + v)}{\left(|c|^2 + uv\right)^2} \tag{5.8}$$

where

$$u = |\zeta|^2 \, , \qquad v = |\xi|^2 \, . \tag{5.9}$$

Noting that

$$\mathcal{E} = -4|c|^2 \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) \left(|c|^2 + uv\right)^{-1} , \tag{5.10}$$

we can easily evaluate the total energy

$$E = 4\pi^2 \int_0^\infty du \int_0^\infty dv \, \mathcal{E}(u, v) \, . \tag{5.11}$$

Introducing a cut-off for small u and v, we can integrate once to get

$$E = 4\pi^2 \left[\int_\delta^\infty dv + \int_\delta^\infty du\right] + \mathcal{O}(\delta) \, . \tag{5.12}$$

Taking the $\delta \to 0$ limit, we recognize this as the formula

$$E = 4\pi A_\xi + 4\pi A_\zeta \tag{5.13}$$

where A_ξ and A_ζ are the (infinite) areas of the ζ and ξ planes, respectively. The coefficient 4π is the lump energy, which equals the area of the unit 2-sphere for the model under discussion; it has an obvious interpretation as the energy per unit area (tension) of a lump-membrane.

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This sigma-model solution is reminiscent of the solution of the $D=5$ supermembrane equations describing a pair of intersecting membranes (see, for example, [8]). In fact, the $D=5$ supermembrane action provides an effective description of the lump-membrane of the $D=5$ supersymmetric sigma model, and so we should expect to find an effective description of the intersecting lump-membranes as a solution of it. However, it should be remembered that the lump-membranes are membranes with a lump core of a definite *non-zero* size (although this size is arbitrary). With this in mind, it is instructive to examine the energy density function $\mathcal{E}(u,v)$ as a function of u for fixed v. Noting that

$$\partial_u \mathcal{E} = \frac{4|c|^2\left(|c|^2 - v^2\right)}{(|c|^2+uv)^3}\,, \tag{5.14}$$

we see that $\mathcal{E}$ is a monotonically decreasing function of u for $v>|c|$ and hence in the limit of large v, but for $v<|c|$ the energy density is a monotonically *increasing* function of u. We conclude that two membranes intersect in a region of size $\sqrt{|c|}$.

Finally, we observe that there is an analogous solution of the massive model but with $\varphi = \pm mt$. We will see later that this solution also preserves 1/4 supersymmetry if $\varphi = mt$, but breaks all supersymmetry if $\varphi = -mt$, despite the time-reversal invariance of the sigma-model equations.

6. Supersymmetry

The condition for a supersymmetric sigma model field configuration to preserve some fraction of the 8 supersymmetries of the sigma-model vacuum is particularly simple for the HK manifolds described in section 2, and can be found in [11, 5]. Since all maximally-supersymmetric sigma models in spacetime dimension $(2+1)$ and above can be obtained by dimensional reduction (trivial or otherwise) of the massless $(5+1)$-dimensional model, we need only consider that case. The number of supersymmetries preserved by any configuration of this theory is the number of linearly-independent *constant* SU(2)-Majorana-Weyl spinor solutions λ to the equation

$$\gamma^\mu\left[\boldsymbol{\tau}\cdot\partial_\mu \mathbf{X} + iU^{-1}\mathcal{D}_\mu\varphi\right]\lambda = 0\,, \tag{6.1}$$

where $\boldsymbol{\tau}$ is the triplet of Pauli matrices. We have supressed both the SO(5, 1) spinor and SU(2) indices. The chirality condition on λ will be taken to be

$$\Gamma^{012345}\lambda = \lambda\,. \tag{6.2}$$

This reduces the 16 complex components of λ to 8 complex components but only 8 real linear combinations are linearly independent because of the SU(2)-Majorana condition (which we do not give here as we shall never need to use its explicit form). The fraction of supersymmetry preserved by configurations of lower-dimensional sigma models can also be determined from (6.1) by lifting them to $(5+1)$ dimensions via the reduction ansatz (2.4) (with $m=0$ for massless models).

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We shall now determine the fraction of supersymmetry preserved by a generic solution of equations (4.7) and (4.8). Using (4.8) to eliminate the space derivatives of φ in the supersymmetry preservation condition (6.1), we find for $m=0$ that

$$(\partial_j X_k)\left[\Gamma^j \tau_k + i\Gamma^i \varepsilon_{ijk} + i\Gamma^4 \delta_{jk}\right]\lambda + (\partial_4 X_i)\left[\Gamma^4 \tau_i - i\Gamma^i\right]\lambda = 0\,, \tag{6.3}$$

where τ_i ($i=1,2,3$) are the three Pauli matrices. For $m \neq 0$ we have the additional condition

$$\Gamma^{05}\lambda = \pm\lambda\,, \tag{6.4}$$

where the choice of sign is inherited from (4.7).

We shall begin our analysis by considering the $m=0$ case. When $\partial_4 \mathbf{X}$ is non-zero we must set

$$\tau_k \lambda = i\Gamma^4 \Gamma^k \lambda \qquad (k=1,2,3)\,. \tag{6.5}$$

These conditions imply that

$$\Gamma^{1234}\lambda = \lambda\,. \tag{6.6}$$

Using (6.5) in (6.3) we find that

$$(\partial_j X_k)\left[\Gamma^4 \Gamma^{jk} - \Gamma^i \varepsilon_{ijk}\right]\lambda = 0\,, \tag{6.7}$$

but this is satisfied identically as a consequence of (6.6). As all three conditions (6.5) are independent we deduce that the generic ($m=0$) solution of the BPS equations (4.8) preserves 1/8 supersymmetry.

Because of the chirality condition (6.2), the constraint (6.6) is equivalent to

$$\Gamma^{05}\lambda = \lambda\,. \tag{6.8}$$

It follows that the additional constraint (6.4) that is needed for preservation of supersymmetry when $m \neq 0$ is identically satisfied if we choose the upper sign, but is otherwise violated. What this means is that there can be 1/8-supersymmetric intersections of Q-lump membranes in the massive $(4+1)$-dimensional theory, but only if $\dot\varphi$ is positive. Reversing the time orientation takes the supersymmetric solution into a non-supersymmetric one. The asymmetry arises from the choice of chirality of the $D=6$ spinor λ and the fact that this asymmetry is maintained by a non-trivial dimensional reduction.

For the special case in which $X_2 = X_3 = 0$, as occurs for the particular solution of section 5, the supersymmetry preservation condition (6.3) becomes

$$\left(\Gamma^4 \partial_1 X_3 - \Gamma^1 \partial_4 X_3\right)\left(1 - i\Gamma^{41}\tau_1\right)\lambda + \left(\Gamma^2 \partial_3 X_3 - \Gamma^3 \partial_2 X_3\right)\left(1 - i\Gamma^{23}\tau_1\right)\lambda = 0\,. \tag{6.9}$$

For generic X_3 this implies the conditions

$$i\Gamma^{23}\tau_1 \lambda = \lambda\,, \qquad i\Gamma^{41}\tau_1 \lambda = \lambda\,, \tag{6.10}$$

which preserve 1/4 supersymmetry. Given the chirality condition (6.2) these conditions imply (6.8), so the stationary, charged, version of the solution of section 5 either preserves 1/4 supersymmetry or breaks all supersymmetries, again depending on the time orientation.

Let us now consider the simpler set of BPS equations (4.9), appropriate to solutions that are independent of x^4. In this case we need not impose the conditions (6.5). Instead, we may impose the conditions

$$i\Gamma_{045}\Gamma^k \tau_k \lambda = \lambda \,. \tag{6.11}$$

This yields

$$(\partial_j X_k)\left[\Gamma^{045}\Gamma^{jk} + \Gamma^i \varepsilon_{ijk}\right]\lambda = 0\,, \tag{6.12}$$

but this is automatically satisfied as a consequence of the chirality condition (6.2). The three new conditions (6.11) are no longer independent because, given the chirality condition, any two imply the third. Thus, generic solutions of (4.6) preserve 1/4 supersymmetry, in agreement with the result of [6]. However, (6.4) is no longer implied by these conditions so we conclude (again in agreement with [6]) that the time-dependent Q-lump version of the 1/4 supersymmetric intersection of sigma-model lump-strings preserves only 1/8 supersymmetry, but does so irrespective of the time orientation.

7. Calibrated M5-branes

Consider an M5-brane in a HK_4 background, as described by the array

$$\begin{array}{lcccccccccc} HK_4: & - & - & - & - & - & 6 & 7 & 8 & \underline{9} & - \\ M5: & 1 & 2 & 3 & 4 & 5 & - & - & - & - & - \end{array} \tag{7.1}$$

The coordinate x^9 is identified with the angular coordinate φ of the HK_4 manifold, and the $6,7,8$ directions with the cartesian coordinates $\mathbf{X}$ on $\mathbb{E}^3$. If the M5-brane is fixed in the remaining tenth direction and has vanishing worldvolume gauge fields then its dynamics will be described at low energy by a supersymmetric $(5+1)$-dimensional sigma model with HK_4 as its target space. As explained earlier, each of the singularities of the function U on $\mathbb{E}^3$ that determines the HK_4 metric will be connected to at least one other singularity by a holomorphic 2-sphere. For simplicity, suppose that there are only two singularities, separated in the 6-direction. Then we may wrap another M5-brane on this 2-sphere. This second M5-brane may smoothly intersect the first one in a 3-brane, yielding a configuration that preserves 1/8 of the 32 supersymmetries of the M-theory vacuum. This configuration is described by the array

$$\begin{array}{lcccccccccc} HK_4: & - & - & - & - & - & 6 & 7 & 8 & \underline{9} & - \\ M5: & 1 & 2 & 3 & 4 & 5 & - & - & - & - & - \\ M5: & - & - & 3 & 4 & 5 & 6 & - & - & 9 & - \end{array} \tag{7.2}$$

Such 3-brane intersections of M5-branes are 1/2 supersymmetric lump solitons of the effective HK sigma model on the first M5-brane [9, 5]. An M5-brane configuration preserving only 1/16 of the supersymmetry of the M-theory vacuum may be obtained by the addition of another M5-brane, according to the array

$$\begin{array}{lcccccccccc} HK_4: & - & - & - & - & - & 6 & 7 & 8 & \underline{9} & - \\ M5: & 1 & 2 & 3 & 4 & 5 & - & - & - & - & - \\ M5: & 1 & - & - & 4 & 5 & 6 & - & - & 9 & - \\ M5: & - & 2 & 3 & - & 5 & 6 & - & - & 9 & - \end{array} \tag{7.3}$$

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As the 5 direction is common to all M5-branes we can trivially compactify it to arrive at an effective $(4+1)$-dimensional sigma model. The array can then be interpreted as the 1/4 supersymmetric intersection of two lump-membranes. As we saw in section 5, this configuration needs only the simplest 2-centre HK_4 metric for its realization.

Now consider the six-centre metric of section 3. We argued in section 4 that this would allow two further pairs of intersecting lump-membranes, leading to a configuration in which six lump-membranes, in the six planes containing two axes, intersections This possibility is associated to the array

$$\begin{array}{l} HK_4: - - - - - 6\ 7\ 8\ \underline{9}\ - \\ M5:\ 1\ 2\ 3\ 4\ 5 - - - - - \\ M5:\ 1 - - 4\ 5\ 6 - - 9 - \\ M5: - 2 - 4\ 5 - 7 - 9 - \\ M5: - - 3\ 4\ 5 - - 8\ 9 - \\ M5:\ 1 - 3 - 5 - 7 - 9 - \\ M5: - 2\ 3 - 5\ 6 - - 9 - \\ M5:\ 1\ 2 - - 5 - - 8\ 9 - \end{array} \tag{7.4}$$

Note that this contains the previous array. Also contained as a sub-array of (7.4) is

$$\begin{array}{l} HK_4: - - - - - 6\ 7\ 8\ \underline{9}\ - \\ M5:\ 1\ 2\ 3\ 4\ 5 - - - - - \\ M5:\ 1 - - 4\ 5\ 6 - - 9 - \\ M5: - 2 - 4\ 5 - 7 - 9 - \\ M5: - - 3\ 4\ 5 - - 8\ 9 - \end{array} \tag{7.5}$$

As the 4 and 5 directions are now common to all M5-branes we can trivially compactify them to arrive at a configuration of three intersecting lump-strings of a $(3+1)$-dimensional sigma model.

Each of the above arrays can be interpreted as describing a single calibrated M5-brane [14, 15]. The array (7.2) corresponds to the Kähler calibration discussed in detail in [10]. The array (7.3) is another Kähler calibration, which was associated to the kink-lump in [5] but has a more direct interpretation in terms of the intersecting lump-membrane solution presented in section 5. The remaining arrays correspond to exceptional calibrations. Specifically, (7.4) corresponds to a Cayley calibrated 4-surface in $\mathbb{E}^4 \times HK_4$, while the sub-array (7.5) corresponds to an associative 3-surface in $\mathbb{E}^3 \times HK_4$. We need discuss only the Cayley calibration in detail as it contains the other cases as sub-cases. We shall see below that the low-energy limit of the equations for a Cayley-calibrated 4-surface in $\mathbb{E}^4 \times HK_4$ yields equations that are equivalent to the sigma-model BPS equation (4.5).

8. M-theory supersymmetry

For any of the calibrations of interest here, the equations that govern the calibrated surface can be obtained from the requirement of partial preservation of supersymmetry by an M5-brane. The number of supersymmetries preserved is the dimension of the space of solutions, for the 32-component covariantly-constant real spinor ϵ, of the condition

$$\Gamma\epsilon = \epsilon\,, \tag{8.1}$$

where Γ is the 'κ-symmetry' matrix of the M5-brane. Let us take the $D = 11$ spacetime coordinates to be $X^M = (X^m, X^\natural)$ where $X^m = (X^\mu, X^I)$ are $D = 10$ spacetime coordinates for $\mathbb{E}^{(1,5)} \times HK_4$. We denote the corresponding $D = 11$ Dirac matrices by

$$\Gamma_M = (\Gamma_\mu, \Gamma_I, \Gamma_\star\Gamma_{6789})\,, \tag{8.2}$$

where

$$\Gamma_\star = \Gamma_{012345}\,. \tag{8.3}$$

For the chosen background we have

$$\{\Gamma_I, \Gamma_J\} = 2G_{IJ} \tag{8.4}$$

where G_{IJ} is the HK_4 metric. The covariantly-constant spinors in this background satisfy

$$\Gamma_{6789}\epsilon = \epsilon \tag{8.5}$$

and take the form $\epsilon = f\epsilon_0$ for some non-zero function f and constant spinor ϵ_0. As the function f cancels we may suppose that the spinor ϵ satisfying (8.1) and (8.5) is constant.

The worldvolume fields of the M5-brane are the maps $X^M(x)$ from the worldvolume, with coordinates (x^μ, x^5) to the $D = 11$ spacetime, and a 3-form self-dual field strength. We set the 3-form field strength to zero and then take $X^\natural$ to be constant; this reduces the worldvolume field content to that of a scalar multiplet of (1,0) $(5+1)$-dimensional supersymmetry, and the matrix Γ to the κ-symmetry matrix of the $D = 10$ $N = 1$ super-5-brane, for which

$$6!\sqrt{-\det g}\,\Gamma = \varepsilon^{\mu\nu\rho\sigma\lambda\eta}\partial_\mu X^m\partial_\nu X^n\partial_\rho X^p\partial_\sigma X^q\partial_\lambda X^r\partial_\eta X^s\Gamma_{mnpqrs}\,. \tag{8.6}$$

In the physical gauge $X^\mu = x^\mu$, the worldvolume metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu X^m\partial_\nu X^n G_{mn}\,, \tag{8.7}$$

and the supersymmetry preserving condition (8.1) becomes

$$(\sqrt{-\det g})\epsilon = \left(1 - \Gamma^\mu\partial_\mu X^I\Gamma_I - \frac{1}{2}\Gamma^{\mu\nu}\partial_\mu X^I\partial_\nu X^J\Gamma_{IJ} + \cdots\right)\Gamma_\star\epsilon\,. \tag{8.8}$$

At zeroth order we have

$$\Gamma_\star\epsilon = \epsilon\,, \tag{8.9}$$

which implies that the vacuum state of the 5-brane is a 1/2 supersymmetric M-theory configuration. Because of the condition (8.5), the background reduces this fraction to 1/4; the surviving 8 supersymmetries are those of the sigma-model vacuum. At first order we have

$$\Gamma^\mu\partial_\mu X^I\Gamma_I\epsilon = 0\,. \tag{8.10}$$

In the special case of Kähler calibrations, this condition actually implies the full M5-brane supersymmetry condition (8.8) [9, 5]. Although this will not be true for calibrations in general, we might expect (8.10) to be equivalent to the sigma model supersymmetry preservation condition (6.1). We shall now verify this.

Introducing a vierbein ${e_I}^A$ for the HK_4 metric, we may write

$$\Gamma_I = {e_I}^A \Gamma_A \tag{8.11}$$

for flat space Dirac matrices $\Gamma_A = (\Gamma_i, \Gamma_\varphi)$. For the HK_4 metric (2.7) the condition (8.10) then becomes

$$\Gamma^\mu \left[\partial_\mu X^i \Gamma_i + U^{-1} \mathcal{D}_\mu \varphi \Gamma_\varphi\right] \epsilon = 0. \tag{8.12}$$

This is equivalent to

$$\Gamma^\mu \left[\partial_\mu \mathbf{X} \cdot \boldsymbol{\sigma} + i U^{-1} \mathcal{D}_\mu \varphi\right] \epsilon = 0\,, \tag{8.13}$$

where we have defined

$$\boldsymbol{\sigma} = i \boldsymbol{\Gamma} \Gamma_\varphi\,. \tag{8.14}$$

Note that, as a consequence of the condition (8.5) imposed by the HK background, these matrices obey the algebra of the quaternions, exactly as do the Pauli matrices $\boldsymbol{\tau}$ appearing in the sigma model preservation condition (6.1). Since the matrices Γ^μ obey the same algebra as the $D = 6$ Dirac matrices γ^μ of (6.1), the implications of (6.1) are the same as those of the linearized supersymmetry preservation condition (8.10), given the additional constraints (8.5) and (8.9) on the $D = 11$ spinor ϵ.

It remains for us to confirm that the BPS equation (4.5) governing the 1/8 supersymmetric intersections of static sigma-model solitons is what is obtained by linearization of the M-theory supersymmetry condition for a Cayley-calibrated M5-brane. For a Cayley calibration in $\mathbb{E}^8$ we have [16, 15], in our conventions

$$(\boldsymbol{\nabla} \cdot \mathbf{X} - \partial_4 \varphi) + \mathbf{i} \cdot (\boldsymbol{\nabla} \times \mathbf{X} - \boldsymbol{\nabla} \varphi - \partial_4 \mathbf{X}) = (dX)^3 \text{ terms} \tag{8.15}$$

where $\mathbf{i}$ are the imaginary units of the quaternions. A solution of these equations can be found in [17]. When the $\mathbb{E}^8$ background is replaced by $\mathbb{E}^4 \times HK_4$, the equation becomes

$$(\boldsymbol{\nabla} \cdot \mathbf{X} - U^{-1} D_4 \varphi) + \mathbf{i} \cdot \left(\boldsymbol{\nabla} \times \mathbf{X} - U^{-1} \boldsymbol{\mathcal{D}} \varphi - \partial_4 \mathbf{X}\right) = (dX)^3 \text{ terms} \tag{8.16}$$

Linearizing in derivatives yields precisely the BPS equation (4.5).

9. Discussion

In this paper we have found a novel BPS equation for intersecting solitonic membranes (lumps) of $(4+1)$-dimensional supersymmetric HK sigma models, for which the generic solution preserves 1/8 supersymmetry. This equation applies to sigma models with HK target spaces of arbitrary quaternionic dimension n, but the $n = 1$ is special because in this case the BPS equation can also be deduced by linearization (in derivatives) of the equations governing Cayley-calibrated 4-surfaces in $\mathbb{E}^4 \times HK_4$, and these calibrations have a physical realization in terms of an M-theory M5-brane. The degeneration to an associative 3-surface in $\mathbb{E}^3 \times HK_4$ yields the equation found in [6] for intersecting lump-strings of the $(3+1)$-dimensional HK sigma model.

We have also shown how a similar result applies to intersecting Q-lumps of the massive $(4+1)$-dimensional sigma-model. A curious feature of this case is that a supersymmetric solution is transformed into a non-supersymmetric one by time reversal! We should stress here that this phenomenon is not just a feature of generic solutions that preserve 1/8 supersymmetry (which remain hypothetical) but is also a feature of an explicit special solution that preserves 1/4 supersymmetry. Although this may seem a strange phenomenon, we remind the reader that a similar phenomenon has long been known for space reversal [18].

We have certainly not exhausted the possible sigma-model soliton intersections in this paper. New possibilities, preserving only 1/8 supersymmetry, are likely to occur in models for which the HK target space has a quaternionic dimension $n \geq 1$, as shown by the multi-domain walls of [19] and the intersecting domain walls of [4], although any M-theory interpretation of these new possibilities will have to be rather different from the $n = 1$ cases discussed here. However, even for $n = 1$ there are likely to be further possibilities; we will conclude with a brief mention of one such case. Let us write the energy functional (4.1) as

$$E = \frac{1}{2}\int d^4x \left\{ |\dot{X} - \sigma m v k|^2 + |\mathbf{I}\cdot\boldsymbol{\nabla} X - m\eta\sqrt{1-v^2}\, k|^2 \right\} + \sigma m v Q + \eta m K \qquad (9.1)$$

where σ and η are two signs, and K is the topological charge

$$K = \int d^4x \, (i_k \boldsymbol{\Omega})_I \cdot \boldsymbol{\nabla} X^I \,. \qquad (9.2)$$

The energy is minimized, for fixed Q and K by solutions of the first-order equations

$$\dot{X} = \sigma m v k \,, \qquad \mathbf{I}\cdot\boldsymbol{\nabla} X = \eta m \sqrt{1-v^2}\, k \,. \qquad (9.3)$$

This equation is solved by the 1/4-supersymmetric Q-kink-lump of [5] but generic solutions preserve 1/8 supersymmetry.

Acknowledgments

It is a pleasure to thank Jerome Gauntlett for helpful discussions. R.P. thanks Trinity College for financial support.

References

[1] P.J. Ruback, *Sigma model solitons and their moduli space metrics*, *Commun. Math. Phys.* **116** (1988) 645.

[2] E.R.C. Abraham and P.K. Townsend, *Q kinks*, *Phys. Lett.* **B 291** (1992) 85; *More on Q-kinks : a* $(1+1)$*-dimensional analogue of dyons*, *Phys. Lett.* **B 295** (1992) 225.

[3] D. Tong, *The moduli space of BPS domain walls*, hep-th/0202012.

[4] J.P. Gauntlett, D. Tong and P.K. Townsend, *Supersymmetric intersecting domain walls in massive hyper- kaehler sigma models*, *Phys. Rev.* **D 63** (2001) 085001 [hep-th/0007124].

[5] J.P. Gauntlett, R. Portugues, D. Tong and P.K. Townsend, *D-brane solitons in supersymmetric sigma-models*, *Phys. Rev.* **D 63** (2001) 085002 [hep-th/0008221].

[6] M. Naganuma, M. Nitta and N. Sakai, *BPS lumps and their intersections in $N = 2$ susy nonlinear sigma models*, hep-th/0108133.

[7] E. Abraham, *Nonlinear sigma models and their q lump solutions*, *Phys. Lett.* **B 278** (1992) 291.

[8] P.K. Townsend, *Brane theory solitons*, hep-th/0004039.

[9] E. Bergshoeff and P.K. Townsend, *Solitons on the Supermembrane*, JHEP 9905:021 (1999)

[10] J. Gutowski, G. Papadopoulos and P.K. Townsend, *Supersymmetry and generalized calibrations*, *Phys. Rev.* **D 60** (1999) 106006 [hep-th/9905156].

[11] J.P. Gauntlett, G.W. Gibbons and P.K. Townsend, *Intersecting domain walls in mqcd*, *Phys. Lett.* **B 483** (2000) 240 [hep-th/0004136].

[12] L. Alvarez-Gaumé and D.Z. Freedman, *Potentials for the supersymmetric nonlinear sigma model*, *Commun. Math. Phys.* **91** (1983) 87.

[13] G.W. Gibbons and S.W. Hawking, *Gravitational multi - instantons*, *Phys. Lett.* **B 78** (1978) 430.

[14] G.W. Gibbons and G. Papadopoulos, *Calibrations and intersecting branes*, *Commun. Math. Phys.* **202** (1999) 593 [hep-th/9803163].

[15] J.P. Gauntlett, N.D. Lambert and P.C. West, *Branes and calibrated geometries*, *Commun. Math. Phys.* **202** (1999) 593.

[16] R. Harvey and H.B. Lawson Jr., *Calibrated geometries*, *Acta Math.* **148** (1982) 47.

[17] D.G. Robertson, *Self-dual lumps and octonions*, *Phys. Lett.* **B 249** (1990) 216.

[18] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *The criterion for vacuum stability in kaluza-klein supergravity*, *Phys. Lett.* **B 139** (1984) 154.

[19] J.P. Gauntlett, D. Tong and P.K. Townsend, *Multi-domain walls in massive supersymmetric sigma-models*, *Phys. Rev.* **D 64** (2001) 025010 [hep-th/0012178].